Lecture Notes *a posteriori* for Math 201

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We defined the tangent space T_pM of a manifold at a point p, and the tangent bundle TM. Zev Choroles gave the following elegant definition of T_pM: We let $\mathcal{D}_p$ be the functions that are differentiable at p. Then an element of T_pM is a linear function $L\colon \mathcal{D}_p \to \mathbb{R}$ such that

$$L(fg) = L(f)g(p) + f(p)L(g).$$

For example, let $M = \mathbb{R}^n$, and let p be the origin. We (essentially) proved that

$$T_pM = \mathbb{R}^n$$

where $(x^1, \ldots, x^n)$ acts on D_0 by

$$D_{(x^1,\ldots,x^n)} f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}.$$

We then observed that if $\mathbf{x}\colon U \to M$ is a chart, and $\mathbf{x}(x) = p$, then we can define $D\mathbf{x}\colon T_x\mathbb{R}^n \to T_pM$ by

$$D\mathbf{x}(v)(f) = v(f \circ \mathbf{x}).$$

This induces an isomorphism from $T_x\mathbb{R}^n \cong \mathbb{R}^n$ to T_pM.

We can then define the tangent bundle TM as the union of T_pM over all $p \in M$. For every chart $\mathbf{x}\colon U \to M$, there is a chart $\hat{\mathbf{x}}\colon U \times \mathbb{R}^n \to M$, defined by $\hat{\mathbf{x}}(x, v) = (\mathbf{x}(x), D\mathbf{x}(v))$. We can easily verify that the overlap maps are smooth, so TM is a smooth manifold.

A *vector field* on M is a smooth map $X\colon M \to TM$ where $X(p) \in T_pM$ for all $p \in M$. Any such vector field defines a map $X\colon C^\infty(M) \to C^\infty(M)$ defined by $(Xf)(p) = X(p)f$. Then $X(fg) = X(f)g + fX(g)$, and any linear map from $C^\infty(M) \to C^\infty(M)$ that satisfies this equation is $f \to Xf$ for some X.

We let $\mathcal{V}$ be the space of all vector fields. We proved (twice) that for all $X, Y \in \mathcal{V}$, there is a unique $Z \in V$ such that $X(Yf) - Y(Xf) = Zf$ for every smooth map f. This defines the Lie bracket $[X, Y]$.

We then observe that for every X there is a unique flow ϕ_t^X defined (for t near 0) on M such that

$$\frac{d\phi_t^X(p)}{dt} = X(\phi_t^X(p)).$$

We then claimed (and did not prove) that

$$[X, Y] = \lim_{t\to 0} \frac{(\phi_t^X)^*Y - Y}{t}.$$

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We proved that

$$[X, Y] = \lim_{t\to 0} \frac{(\phi_t^X)^*Y - Y}{t}. \tag{1}$$

but first we spent a large part of the class defining g^*Y (and g_*Y) for a diffeomorphism $g\colon M \to M$. Suffice it to say that $g^*Y(p) = (Dg)_p^{-1}(Y(p))$. We also observed that if X and Y are vector fields in $\mathbb{R}^n$, then $[X, Y] = D_XY - D_YX$.

To prove equation **??**, we *work in* $\mathbb{R}^n$ and write

$$\begin{aligned}(\phi_t^X) * Y - Y = (\phi_t^X)^*Y - Y \circ \phi_t^X \\ + Y \circ \phi_t^X - Y,\end{aligned}$$

where by $Y \circ \phi$ we mean the vector field defined by $(Y \circ \phi)(p) = Y(g(p))$. *We can only do this in* $\mathbb{R}^n$, because in general, we have no way of translating the vector at $g(p)$ to the vector at p, without using the derivative $(Dg)_p$. We then write

$$(\phi_t^X)^*Y - Y \circ \phi = \big((D\phi_t^X)^{-1} - \mathbf{1}\big) \cdot (Y \circ \phi_t^X),$$

and therefore

$$\begin{aligned}
\lim_{t\to 0} \frac{(\phi_t^X)^* Y - Y\circ\phi}{t} &= \lim_{t\to 0} \frac{\big((D\phi_t^X)^{-1} - \mathbf{1}\big)(Y\circ\phi_t^X)}{t} \\
&= \lim_{t\to 0} \frac{(D\phi_t^X)^{-1} - \mathbf{1}}{t} \cdot \lim_{t\to 0}(Y\circ\phi_t^X) \\
&= \frac{d}{dt}\bigg|_{t=0} (D\phi_t^X)^{-1}\cdot Y \\
&= \frac{d}{dt}\bigg|_{t=0} \big(D\phi_{-t}^X\big)\cdot Y \\
&= -\frac{d}{dt}\bigg|_{t=0} \big(D\phi_t^X\big)\cdot Y \\
&= D\left(-\frac{d}{dt}\bigg|_{t=0} \big(\phi_t^X\big)\right)\cdot Y \\
&= -DX\cdot Y \\
&= -D_Y X.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\lim_{t\to 0} \frac{Y\circ\phi_t^X - Y}{t} &= \frac{d}{dt}\bigg|_{t=0} Y\circ\phi_t^X \\
&= D_X Y
\end{aligned}$$

So

$$\lim_{t\to 0} \frac{(\phi_t^X)^* Y - Y\circ\phi}{t} = -D_Y X + D_X Y = [X,Y].$$

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We defined a Riemannian metric on a smooth manifold M: it is a choice of inner product $\langle\cdot,\cdot\rangle_p$ on T_pM for each point $p\in M$, such that $\langle X(p), Y(p)\rangle_p$ is a smooth function of p whenever X and Y are smooth vector fields on M. We will also write the function $\langle X, Y\rangle \equiv \langle X(p), Y(p)\rangle_p$ as $g(X,Y)$, and we will refer to g as the *metric tensor*: it is a symmetric 2-tensor (bilinear form) that is everywhere positive definite (which means that $g_p(v,v) > 0$ for any nonzero $v \in T_pM$).

If $x_\alpha \colon U_\alpha \to M$ is a chart, we can define a (Riemannian) metric on U_α by

$$\langle v, w \rangle_x = \langle dx_\alpha(v), dx_\alpha(w) \rangle_{x_\alpha(x)}.$$

If we let $g_{ij}(x) = \left\langle \frac{d}{dx^i}, \frac{d}{dx^j} \right\rangle_x$, then

$$\langle V(x), W(x) \rangle_x = g_{ij} V^i W^j,$$

where $V = V^i \frac{d}{dx^i}$ and $W = W^i \frac{d}{dx^i}$ and we follow the "Einstein convention" of summing over indices that appear as both subscripts and superscripts. (Note that the i in $\frac{d}{dx^i}$ appears as a subscript, because it is in a superscript in the denominator).

The standard inner product is a Riemannian metric on $\mathbb{R}^n$: if $v, w \in T_x\mathbb{R}^n$, then we let $\langle v, w \rangle_x = v \cdot w$, where we have identified $T_x\mathbb{R}^n$ with $\mathbb{R}^n$, and $v \cdot w = \sum_i v_i w_i$ is the standard inner product on $\mathbb{R}^n$. If $f \colon M \to N$ is an immersion, we can pull back a Riemannian metric on N to a Riemannian metric on M, by writing $\langle v, w \rangle_p = \langle df_p(v), df(p) \rangle_{f(p)}$. In particular, if $N = \mathbb{R}^n$, we can pull back the Euclidean Riemannian metric (defined above) to M. Thus we have defined a Riemannian metric on every imbedded (or immersed) manifold in $\mathbb{R}^n$.

We observed that the following are equivalent (for an n-manifold M):

1. There is a nonvanishing n-form ω on M.

2. We can find charts $x_\alpha \colon U_\alpha \to M$ such that $\det D(x_\beta^{-1} \circ x_\alpha) > 0$ wherever it is defined.

3. For each point $p \in M$, we can define a function Sgn_p that maps bases of T_pM to 1 and -1, such that if $X_1, \dots X_n$ are (smooth) vector fields defined on an open subset U of M, and $X_1(p), \dots X_n(p)$ form a basis of $T_p(M)$ for every $p \in M$, then $\mathrm{Sgn}_p(X_1(p), \dots, X_n(p))$ is a continuous function of p (and hence constant if U is connected).

If any of these three hold we say that M is orientable, and we call the choice we make in 1, 2, or 3 (the n-form, the charts, or the function Sgn) an orientation for M. Note that there are only two choices for the function Sgn on any orientable manifold M. (There are two choices even if M is a point).

We also observed that if $v_1, \dots v_n$ are n vectors in $\mathbb{R}^n$, then the volume of the parallelpiped they span is given by

$$\det(v_1, \dots, v_n),$$

and

$$\det(a_{ij}) = (\det(v_1, \dots, v_n))^2,$$

where $a_{ij} = \langle v_i, v_j \rangle$. Therefore(?!) we can define a volume form dV on an oriented n-dimensional Riemannian manifold M by

$$dV_p(v_1, \dots v_n) = \mathrm{Sgn}_p(v_1, \dots v_n)\sqrt{\det(a_{ij})},$$

where $a_{ij} = \langle v_i, v_j \rangle_p$. If $x_\alpha \colon U_\alpha \to M$ is an orientation preserving chart for M, then

$$x_\alpha^* dV = \sqrt{\det(g_{ij})}\, dx^1 \ \dots,\ dx^n.$$

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Let M and N be k- and l-manifolds with Riemannian metrics, and suppose that $g \colon M \to N$ is a submersion. Then for any p in M, we let $W_p^g M$ be the orthogonal complement of the kernel of $Dg_p \colon T_pM \to T_{g(p)}N$. Then $W_p^g M$ is a subspace of T_pM of dimension l. We say that g is a *Riemannian submersion* if $Dg_p \colon W_p^g M \to T_{g(p)}M$ is an isometry.

Now let a Lie group G act on M freely and properly discontinuously by isometries. There is a unique metric on M/G for which the projection map $\pi \colon M \to M/G$ is a Riemannian submersion. We can define it by observing that π is a submersion, defining $W_p^\pi M$ as above, and defining, for $q = \pi(p)$,

$$\langle v, w \rangle_q = \left\langle D\pi|_{W_p^\pi M}^{-1} v, D\pi|_{W_p^\pi M}^{-1} w \right\rangle.$$

This will be well-defined (independent of the choice of q) because $\pi(p) = \pi(p')$ implies that $p' = g \cdot p$ for some $g \in G$, and $Dg_p \colon W_p^\pi M \to W_{p'}^\pi M$ is an isometry.

One application of this is the *Fubini-Study* metric on $\mathbb{C}P^n$. We let $S^1 = U(1)$ act on $S^{2n+1} \subset \mathbb{C}^{n+1}$ by $\omega \cdot (z_0, \dots, z_n) = (\omega z_0, \dots, \omega z_n)$. Then the quotient is $\mathbb{C}P^n$, and it has a unique metric for which the projection from $S^{2n+1} \to \mathbb{C}P^n$ is an isometry.

We say that a Riemannian manifold M is a *homogeneous space* if the group of isometries of M is transitive: for every $x, y \in M$, we can find an isometry $g\colon M \to M$ such that $g(x) = y$. The group of isometries of M is always a Lie group, and we can write

$$M = \operatorname{Isom} M / \operatorname{Stab}_p,$$

where $\operatorname{Stab}_p < \operatorname{Isom} M$ is the group of isometries stabilizing a point p.

Question 1 *Suppose that M, g_0 is a homogeneous space. What is the space of Riemannian metrics g_1 on M that is invariant under* $\operatorname{Isom} M$*? What conditions on* $\operatorname{Isom} M$ *are sufficient to insure that g_1 is a (constant) rescaling of g_0?*

Exercise 1 *Prove that $\mathbb{C}P^n$ is a homogeneous space (with the Fubini-Study metric) and find the stabilizer of a point.*

We say that M is *isotropic* if M for every $p, q \in M$, and any two orthonormal bases $(x_1, \ldots x_n)$, $(y_1, \ldots, y_n)$ of T_pM and T_qM respectively, we can find an isometry $\phi\colon M \to M$ such that $\phi(p) = q$, and $D\phi(x_i) = y_i$ for $i = 1, \ldots, n$.

Here are the three standard examples of isotropic spaces:

1. $\mathbb{R}^n$ with the standard metric

2. S^n with the standard metric, taken from the standard metric on R^{n+1}.

3. The hyperbolic space H^n. We can define H^n as follows. We consider the pseudo-metric on $\mathbb{R}^{n+1}$ given by

$$\big\langle (x^0, \ldots, x^n), (y^0, \ldots, y^n) \big\rangle = -x^0y^0 + x^1y^1 + \ldots + x^ny^n.$$

   We then restrict this pseudo-metric to the hyperboloid $H^n \subset \mathbb{R}^{n+1}$, defined by $\langle \mathbf{x}, \mathbf{x} \rangle = -1$, and $x^0 > 0$ (where $\mathbf{x} = (x^0, \ldots, x^n)$). We leave it to the reader to check that the restricted pseudo-metric is positive definite, and that H^n is isotropic in this metric.

We will prove, some time in this class, that these are the only examples of isotropic spaces, up to isometry and rescaling the metric.